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# RANDOM ATTRACTORS FOR NON-AUTONOMOUS FRACTIONAL STOCHASTIC PARABOLIC EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. We study the asymptotic behavior of a class of non-autonomous non-local fractional stochastic parabolic equation driven by multiplicative white noise on the entire space $\mathbb{R}^n$. We first prove the pathwise well-posedness of the equation and define a continuous non-autonomous cocycle in $L^2(\mathbb{R}^n)$. We then prove the existence and uniqueness of tempered pullback attractors for the cocycle under certain dissipative conditions. The periodicity of the tempered attractors is also proved when the deterministic non-autonomous external terms are periodic in time. The pullback asymptotic compactness of the cocycle in $L^2(\mathbb{R}^n)$ is established by the uniform estimates on the tails of solutions for sufficiently large space and time variables.

1. **Introduction.** This paper is concerned with the asymptotic behavior of solutions of the following non-autonomous, non-local, fractional stochastic equations on $\mathbb{R}^n$:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t,x,u) + g(t,x) + \alpha u \circ \frac{dW}{dt}, \quad x \in \mathbb{R}^n, \quad t > \tau, \tag{1.1}$$

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\lambda$ and $\alpha$ are positive constants, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $W$ is a two-sided real-valued Wiener process on a probability space, and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity. Note that the stochastic equation (1.1) is understood in the sense of Stratonovich's integration.

The operator $(-\Delta)^s$ is referred to as the fractional Laplacian with $s \in (0,1)$. The differential equations involving the fractional Laplacian have a wide range of applications in physics, biology, chemistry and other fields of science, see [1, 25, 29, 30, 31, 33, 36]. The solutions of fractional deterministic equations have been studied in many publications, see [1, 7, 18, 21, 25, 29, 30, 31, 33, 36, 38, 41, 42, 44, 46, 47], and the references therein. The goal of the present paper is to investigate

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the existence of solutions and the existence of random attractors of the fractional stochastic parabolic equation (1.1)-(1.2).

It is worth mentioning that the random attractors of stochastic equations with the standard Laplace operator (i.e., $s = 1$) have already been examined by many authors. In this respect, the reader is referred to [4, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 23, 24, 26, 32, 34, 43, 45, 48, 50, 51] for the autonomous stochastic equations, and to [2, 13, 19, 27, 28, 35, 52, 53] for the non-autonomous stochastic ones. In the contrast, there are only a few papers in the literature dealing with the existence of random attractors for fractional stochastic equations [38, 39, 40, 54]. More precisely, the authors of [38, 39, 40] discussed the existence of random attractors for stochastic equations involving $(-\Delta)^s$ with $s \in (\frac{1}{2}, 1)$, and the author of [54] proved the existence of such attractors for parabolic equations with $s \in (0, 1)$ defined in bounded domains. There is no result available in the literature for the existence of random attractors for the fractional stochastic equation (1.1) with $s \in (0, 1)$ in the unbounded domains, including the whole space $\mathbb{R}^n$. The purpose of the present paper is to close this gap and prove problem (1.1)-(1.2) has a unique tempered pullback random attractor for all $s \in (0, 1)$ in $L^2(\mathbb{R}^n)$.

The main difficulty of this paper lies in the non-compactness of Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ with $s > 0$ due to the unboundedness of $\mathbb{R}^n$, which introduces a major obstacle for establishing the pullback asymptotic compactness of the solution operator. We overcome this difficulty by using the method of uniform estimates on the tails of solutions [49]. More precisely, for every $\varepsilon > 0$, we show that there exists a large open ball $\mathcal{O}_K$ in $\mathbb{R}^n$ with center at origin and radius $K > 0$ such that the solutions are uniformly less than $\frac{1}{4}\varepsilon$ in $L^2(\mathbb{R}^n \setminus \mathcal{O}_K)$ when time is sufficiently large. Since $\mathcal{O}_K$ is bounded and the embedding $H^s(\mathcal{O}_K) \hookrightarrow L^2(\mathcal{O}_K)$ is compact with $s > 0$, by the uniform estimates, we can prove that the solutions are compact in $L^2(\mathcal{O}_K)$. Consequently, the solutions are covered by a finite number of open balls in $L^2(\mathcal{O}_K)$ with radii less than $\frac{1}{4}\varepsilon$. This along with the uniform tail-estimates implies that the solutions are covered by a finite number of open balls in $L^2(\mathbb{R}^n)$ with radii less than $\varepsilon$, and hence the solutions are asymptotically compact in $L^2(\mathbb{R}^n)$, see Lemma 5.4 for more details. Compared with the equations with standard Laplace operator, the uniform estimates on the tails of solutions are much more involved because of the non-local nature of the fractional Laplace operator $(-\Delta)^s$, see Lemma 4.4 in Section 4.

The rest of the paper is organized as follows. In Section 2, we review some basic results on the existence of random attractors for non-autonomous random dynamical systems. In Section 3, we prove the pathwise well-posedness of problem (1.1)-(1.2) in $L^2(\mathbb{R}^n)$ and define a continuous non-autonomous cocycle over a metric dynamical system. The uniform estimates of solutions are contained in Section 4, and the proof of existence of tempered random attractors is given in Section 5.

## 2. **Preliminaries.**

In this section, we briefly review some notations and results for non-autonomous random dynamical systems for the sake of readers' convenience. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $(X, d)$ is a separable metric space. We use $d(A, B)$ to denote the Hausdorff semi-distance for nonempty subsets $A$ and $B$ of $X$.

**Definition 2.1.** Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical systems. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

(i) $\Phi(\cdot,\tau,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping;
(ii) $\Phi(0,\tau,\omega,\cdot)$ is the identity on $X$;
(iii) $\Phi(t+s,\tau,\omega,\cdot) = \Phi(t,\tau+s,\theta_s\omega,\cdot) \circ \Phi(s,\tau,\omega,\cdot)$;
(iv) $\Phi(t,\tau,\omega,\cdot) : X \to X$ is continuous.

In addition, if there exists a positive number $T$ such that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t,\tau+T,\omega,\cdot) = \Phi(t,\tau,\omega,\cdot),$$

then $\Phi$ is called a continuous periodic cocycle on $X$ with periodic $T$.

**Definition 2.2.** Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then $\Phi$ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any sequences $t_n \to +\infty$, $x_n \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in X.}$$

**Definition 2.3.** Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor of $\Phi$ if the following conditions are satisfied:

(i) $\mathcal{A}$ is measurable and $\mathcal{A}(\tau,\omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
(ii) $\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega), \quad t \geq 0;$$

(iii) $\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, given $B \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} d(\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

In addition, if there exists $T > 0$ such that

$$\mathcal{A}(\tau+T,\omega) = \mathcal{A}(\tau,\omega), \quad \forall\, \tau \in \mathbb{R},\ \omega \in \Omega,$$

then we say $\mathcal{A}$ is periodic with period $T$.

The following results can be found in [52, 53] (see also [16, 17, 45] for related results).

**Proposition 2.4.** *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and $\Phi$ be a continuous cocycle on X over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. If $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in X and has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then $\Phi$ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau,\omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t,\tau-t,\theta_{-t}\omega, K(\tau-t,\theta_{-t}\omega))}.$$

For the periodicity of $\mathcal{D}$-pullback attractors, we have the following proposition from [52].

**Proposition 2.5.** *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X. Suppose $\Phi$ is a continuous periodic cocycle with period $T > 0$ on X over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. If $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in X and has a closed measurable T-periodic $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then $\Phi$ has a unique T-periodic $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$.*

Next, we recall some notations related to the fractional derivatives and fractional Sobolev spaces. Given $0 < s < 1$, the fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy, \quad x \in \mathbb{R}^n,$$

provided the integral exists, where $C(n,s)$ is a positive constant depending on $n$ and $s$ as given by

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi_1)}{|\xi|^{n+2s}}d\xi\right)^{-1}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n. \tag{2.1}$$

It follows from [18] that

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)), \quad \xi \in \mathbb{R}^n,$$

where $\mathcal{F}$ is the Fourier transform. Let $H^s(\mathbb{R}^n)$ be the fractional Sobolev space defined by

$$H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy < \infty\right\}$$

with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{\frac{1}{2}}.$$

Throughout this paper, we denote the norm and the inner product of $L^2(\mathbb{R}^n)$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. For convenience, the Gagliardo semi-norm of $H^s(\mathbb{R}^n)$ is denoted $\|\cdot\|_{\dot{H}^s(\mathbb{R}^n)}$, i.e.,

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy, \quad u \in H^s(\mathbb{R}^n).$$

We also use the notation

$$(u,v)_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy, \quad u,\ v \in H^s(\mathbb{R}^n).$$

Then for all $u \in H^s(\mathbb{R}^n)$ we have $\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2 + \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}$. Note that $H^s(\mathbb{R}^n)$ is a Hilbert space with inner product given by

$$(u,v)_{H^s(\mathbb{R}^n)} = (u,v) + (u,v)_{\dot{H}^s(\mathbb{R}^n)}, \quad u,\ v \in H^s(\mathbb{R}^n).$$

By [18], we have

$$\|(-\Delta)^{\frac{s}{2}}u\|^2 = \frac{1}{2}C(n,s)\|u\|^2_{\dot{H}^s(\mathbb{R}^n)}, \quad \text{for all } \ u \in H^s(\mathbb{R}^n),$$

and hence

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2 + \frac{2}{C(n,s)}\|(-\Delta)^{\frac{s}{2}}u\|^2, \quad \text{for all } \ u \in H^s(\mathbb{R}^n).$$

This implies that $\left(\|u\|^2 + \|(-\Delta)^{\frac{1}{2}}u\|^2\right)^{\frac{1}{2}}$ is an equivalent norm of $H^s(\mathbb{R}^n)$.

3. **Cocycles.** In this section, we establish the existence of a continuous cocycle for the following non-autonomous fractional stochastic equation with $s \in (0,1)$:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t,x,u) + g(t,x) + \alpha u \circ \frac{dW}{dt}, \quad x \in \mathbb{R}^n, \quad t > \tau, \tag{3.1}$$

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \tag{3.2}$$

where $\lambda$ and $\alpha$ are positive constants, $g \in L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$, $W$ is a two-sided real-valued Wiener process on a probability space. The nonlinearity $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies, for all $t, u \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t,x,u)u \le -\beta |u|^p + \psi_1(t,x), \tag{3.3}$$

$$|f(t,x,u)| \le \psi_2(t,x)|u|^{p-1} + \psi_3(t,x), \tag{3.4}$$

$$\frac{\partial f}{\partial u}(t,x,u) \le \psi_4(t,x), \tag{3.5}$$

where $\beta > 0$ and $p \ge 2$ are constants,

$$\psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^n)), \ \ \psi_2, \psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)), \ \ \psi_3 \in L^q_{loc}(\mathbb{R}, L^q(\mathbb{R}^n))$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $(\Omega, \mathcal{F}, P)$ be the standard probability space where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $P$ is the Wiener measure on $(\Omega, \mathcal{F})$. Denote by $\theta_t : \Omega \to \Omega$ the transformation

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system. Consider the one-dimensional stochastic equation:

$$dy + ydt = dW.$$

It follows from [3] that this equation has a unique stationary solution $y(t) = z(\theta_t \omega)$ where $z : \Omega \to \mathbb{R}$ is a random variable given by $z(\omega) = -\int_{-\infty}^0 e^\tau \omega(\tau) d\tau$ for $\omega \in \Omega$. Moreover, there exists a $\theta_t$-invariant set of full measure $\Omega_0$ such that $z(\theta_t \omega)$ is pathwise continuous for every $\omega \in \Omega_0$ and

$$\lim_{t \to \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0 \quad \text{and} \quad \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z(\theta_t \omega) dt = 0. \tag{3.6}$$

For convenience, in the sequel, we will not distinguish $\Omega_0$ and $\Omega$ and use the same notation $\Omega$ for both $\Omega_0$ and $\Omega$.

For our purpose, we need to convert the stochastic equation (3.1) into a deterministic one parametrized by $\omega \in \Omega$. To that end, we introduce a new variable $v = v(t, \tau, \omega, v_\tau)$ by

$$v(t, \tau, \omega, v_\tau) = e^{-\alpha z(\theta_t \omega)} u(t, \tau, \omega, u_\tau) \quad \text{with} \quad v_\tau = e^{-\alpha z(\theta_\tau \omega)} u_\tau, \tag{3.7}$$

where $\tau \in \mathbb{R}$ is a deterministic initial time, $t \ge \tau$, $\omega \in \Omega$, $u_\tau \in L^2(\mathbb{R}^n)$, and $u = u(t, \tau, \omega, u_\tau)$ is a solution of (3.1)-(3.2). Then we find that for $t > \tau$,

$$\frac{dv}{dt} + (-\Delta)^s v + \lambda v = \alpha z(\theta_t \omega) v + e^{-\alpha z(\theta_t \omega)} f(t, x, e^{\alpha z(\theta_t \omega)} v) + e^{-\alpha z(\theta_t \omega)} g(t,x), \ x \in \mathbb{R}^n, \tag{3.8}$$

with initial condition

$$v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n. \tag{3.9}$$

To define a continuous cocycle for the fractional stochastic equation (3.1), we first need to prove the existence and uniqueness of solutions of problem (3.8)-(3.9).

By a solution $v$ of (3.8)-(3.9), we mean $v$ satisfies the equation in the following sense.

**Definition 3.1.** Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathbb{R}^n)$, a continuous function $v(\cdot, \tau, \omega, v_\tau)$: $[\tau, \infty) \to L^2(\mathbb{R}^n)$ is called a solution of problem (3.8)-(3.9) if $v(\tau, \tau, \omega, v_\tau) = v_\tau$ and

$$v \in L^2_{loc}((\tau, \infty), H^s(\mathbb{R}^n)) \bigcap L^p_{loc}((\tau, \infty), L^p(\mathbb{R}^n)),$$

$$\frac{dv}{dt} \in L^2_{loc}((\tau, \infty), H^{-s}(\mathbb{R}^n)) + L^q_{loc}((\tau, \infty), L^q(\mathbb{R}^n)),$$

and $v$ satisfies, for every $\xi \in H^s(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$,

$$\frac{d}{dt}(v, \xi) + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(\xi(x) - \xi(y))}{|x - y|^{n+2s}} dxdy + \lambda(v, \xi)$$
$$= \alpha z(\theta_t \omega)(v, \xi) + e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} f(t, x, e^{\alpha z(\theta_t \omega)} v) \xi(x) dx + e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} g(t, x) \xi(x) dx \tag{3.10}$$

in the sense of distribution on $(\tau, \infty)$.

To prove the existence of solutions of (3.8)-(3.9) in the sense of Definition 3.1, we will approximate the entire space $\mathbb{R}^n$ by a bounded domain $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$ and then take the limit as $k \to \infty$. Let $\rho : [0, \infty) \to \mathbb{R}$ be a smooth function such that $0 \le \rho(s) \le 1$ for all $0 \le s < \infty$ and

$$\rho(s) = 1 \quad \text{for} \quad 0 \le s \le \frac{1}{2} \quad \text{and} \quad \rho(s) = 0 \quad \text{for} \quad s \ge 1. \tag{3.11}$$

Consider the following non-autonomous fractional equation on $\mathcal{O}_k$:

$$\begin{aligned} \frac{dv_k}{dt} + (-\Delta)^s v_k + \lambda v_k =& \alpha z(\theta_t \omega) v_k + e^{-\alpha z(\theta_t \omega)} f(t, x, e^{\alpha z(\theta_t \omega)} v_k) \\ &+ e^{-\alpha z(\theta_t \omega)} g(t, x), \ x \in \mathcal{O}_k, \ t > \tau \end{aligned} \tag{3.12}$$

with boundary condition

$$v_k(t, x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}_k, \ t > \tau, \tag{3.13}$$

and initial condition

$$v_k(\tau, x) = \rho\left(\frac{|x|}{k}\right) v_\tau(x), \quad x \in \mathcal{O}_k, \tag{3.14}$$

where $v_\tau \in L^2(\mathbb{R}^n)$. Note that in the boundary condition (3.13), we require $v_k = 0$ on the complement of $\mathcal{O}_k$ (i.e., on $\mathbb{R}^n \setminus \mathcal{O}_k$), not just on the boundary of $\mathcal{O}_k$. This boundary condition is consistent with the definition of the non-local fractional operator $(-\Delta)^s$. To present the existence of solutions of problem (3.12)-(3.14), for every $k \in \mathbb{N}$, we set $H_k = \{v \in L^2(\mathbb{R}^n) : \ v = 0 \text{ a.e. for } |x| \ge k\}$ and $V_k = \{v \in H^s(\mathbb{R}^n) : \ v = 0 \text{ a.e. for } |x| \ge k\}$. The dual space of $V_k$ is denoted $V_k^*$.

Let $a : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be a bilinear form given by, for $v_1, v_2 \in H^s(\mathbb{R}^n)$,

$$a(v_1, v_2) = \lambda(v_1, v_2) + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_1(x) - v_1(y))(v_2(x) - v_2(y))}{|x - y|^{n+2s}} dxdy. \tag{3.15}$$

By using the bilinear form $a$, we define $A$: $H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ by

$$(A(v_1), v_2)_{(H^{-s}, H^s)} = a(v_1, v_2) \quad \text{for all } v_1, v_2 \in H^s(\mathbb{R}^n), \tag{3.16}$$

where $(\cdot,\cdot)_{(H^{-s},H^s)}$ is the duality pairing of $H^{-s}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. Since $H_k$ and $V_k$ are subspaces of $L^2(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$, respectively, we find that $a: V_k\times V_k\to\mathbb{R}$ and $A$: $V_k\to V_k^*$ are well defined. Indeed, we have

$$(A(v_1),v_2)_{(V_k^*,V_k)}=a(v_1,v_2)\quad\text{for all } v_1,v_2\in V_k,$$

where $(\cdot,\cdot)_{(V_k^*,V_k)}$ is the duality pairing of $V_k^*$ and $V_k$. Under conditions (3.3)-(3.5), it follows from [54] that for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $v_\tau\in L^2(\mathbb{R}^n)$, problem (3.12)-(3.14) has a unique solution $v_k$ in the sense that $v_k(\cdot,\tau,\omega,v_\tau)$: $[\tau,\infty)\to H_k$ is continuous, $v_k(\tau,\tau,\omega,v_\tau)(x)=\rho(\frac{|x|}{k})v_\tau(x)$ and

$$v_k\in L^2_{loc}((\tau,\infty),V_k)\bigcap L^p_{loc}((\tau,\infty),L^p(\mathbb{R}^n)),$$

$$\frac{dv_k}{dt}\in L^2_{loc}((\tau,\infty),V_k^*)+L^q_{loc}((\tau,\infty),L^q(\mathbb{R}^n)),$$

and $v_k$ satisfies, for every $\xi\in V_k\bigcap L^p(\mathbb{R}^n)$,

$$\begin{aligned}&\frac{d}{dt}(v_k,\xi)+\frac12 C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v_k(x)-v_k(y))(\xi(x)-\xi(y))}{|x-y|^{n+2s}}dxdy+\lambda(v_k,\xi)\\&=\alpha z(\theta_t\omega)(v_k,\xi)+e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}_k}f(t,x,e^{\alpha z(\theta_t\omega)}v_k)\xi(x)dx\\&+e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}_k}g(t,x)\xi(x)dx\end{aligned}\tag{3.17}$$

in the sense of distribution on $(\tau,\infty)$. Next, we derive uniform estimates of the solution $v_k$ with respect to $k\in\mathbb{N}$ and prove the existence of solutions of (3.8)-(3.9) by taking the limit of $v_k$ when $k\to\infty$.

**Theorem 3.2.** *Suppose (3.3)-(3.5) hold. Then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $v_\tau\in L^2(\mathbb{R}^n)$, problem (3.8)-(3.9) has a unique solution $v(t,\tau,\omega,v_\tau)$ in the sense of Definition 3.1. This solution is $(\mathcal{F},\mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega$ and continuous in initial data $v_\tau$ in $L^2(\mathbb{R}^n)$. Moreover, the solution $v$ satisfies the energy equation:*

$$\begin{aligned}&\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2+C(n,s)\|v\|^2_{\dot H^s}+2\lambda\|v\|^2\\&=2\alpha z(\theta_t\omega)\|v\|^2+2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}f(t,x,e^{\alpha z(\theta_t\omega)}v)vdx+2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}g(t,x)vdx\end{aligned}\tag{3.18}$$

*for almost all $t\ge\tau$.*

*Proof.* The proof is similar to the case of bounded domains as in [54]. Of course, for problem (3.8)-(3.9) defined on the unbounded domain $\mathbb{R}^n$, we must show that all estimates on the solutions of (3.12)-(3.14) are uniform with respect to all $k\in\mathbb{N}$.

**Step (i). Uniform estimates of solutions of (3.12)-(3.14).** By (3.12) we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathcal{O}_k}|v_k(x)|^2dx+\int_{\mathcal{O}_k}v_k(x)(-\Delta)^s v_k(x)dx+\lambda\int_{\mathcal{O}_k}|v_k(x)|^2dx\\&=\alpha z(\theta_t\omega)\int_{\mathcal{O}_k}|v_k(x)|^2dx+e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}_k}f(t,x,e^{\alpha z(\theta_t\omega)}v_k)v_kdx\\&+e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}_k}g(t,x)v_k(x)dx.\end{aligned}$$

By the boundary condition (3.13), all above integrals over the bounded domain $\mathcal{O}_k$ can be replaced by that over the entire space $\mathbb{R}^n$, and hence we get

$$\frac{1}{2}\frac{d}{dt}\|v_k\|^2 + a(v_k, v_k) = \alpha z(\theta_t\omega)\|v_k\|^2 + e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} f(t, x, e^{\alpha z(\theta_t\omega)}v_k)v_k dx + e^{-\alpha z(\theta_t\omega)}(g(t), v_k). \tag{3.19}$$

By (3.3), the nonlinear term in (3.19) satisfies

$$e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} f(t, x, e^{\alpha z(\theta_t\omega)}v_k)v_k dx \le -\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v_k|^p dx + e^{-2\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\psi_1(t, x)dx. \tag{3.20}$$

It follows from (3.15) and (3.19)-(3.20) that for every $k \in \mathbb{N}$,

$$\frac{d}{dt}\|v_k\|^2 + 2\lambda\|v_k\|^2 + C(n, s)\|v_k\|^2_{\dot{H}^s(\mathbb{R}^n)} + 2\beta e^{(p-2)\alpha z(\theta_t\omega)}\|v_k\|^p_{L^p(\mathbb{R}^n)} \le (1 + 2\alpha z(\theta_t\omega))\|v_k\|^2 + 2e^{-2\alpha z(\theta_t\omega)}\|\psi_1(t)\|_{L^1(\mathbb{R}^n)} + e^{-2\alpha z(\theta_t\omega)}\|g(t)\|^2. \tag{3.21}$$

By (3.21) and (3.16) we see that for every fixed $\omega \in \Omega$ and $T > 0$, $\{v_k\}_{k=1}^{\infty}$ is bounded in

$$L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \bigcap L^2(\tau, \tau + T; H^s(\mathbb{R}^n)) \bigcap L^p(\tau, \tau + T; L^p(\mathbb{R}^n)) \tag{3.22}$$

and

$$\{A(v_k)\}_{k=1}^{\infty} \text{ is bounded in } L^2(\tau, \tau + T; H^{-s}(\mathbb{R}^n)). \tag{3.23}$$

By (3.4) and (3.22) one can verify that

$$\{f(t, \cdot, e^{\alpha z(\theta_t\omega)}v_k)\}_{k=1}^{\infty} \text{ is bounded in } L^q(\tau, \tau + T; L^q(\mathbb{R}^n)). \tag{3.24}$$

As a consequence of (3.12) and (3.23)-(3.24) we find that for each fixed $K \in N$,

$$\left\{\frac{dv_k}{dt}\right\}_{k=1}^{\infty} \text{ is bounded in } L^q(\tau, \tau + T; (V_K \bigcap L^p(\mathbb{R}^n))^*). \tag{3.25}$$

Note that $1 < q \le 2$ since $p \ge 2$ and $p$ and $q$ are conjugate exponents.

**Step (ii). Existence of solutions of problem (3.8)-(3.9).** By a diagonal process, from (3.22)-(3.24), we find that there exists $\widetilde{v} \in L^2(\mathbb{R}^n)$,

$$v \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \bigcap L^2(\tau, \tau + T; H^s(\mathbb{R}^n)) \bigcap L^p(\tau, \tau + T; L^p(\mathbb{R}^n))$$

and $\chi \in L^q(\tau, \tau + T; L^q(\mathbb{R}^n))$ such that, up to a subsequence,

$$v_k \to v \text{ weak-star in } L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)), \tag{3.26}$$

$$v_k \to v \text{ weakly in } L^2(\tau, \tau + T; H^s(\mathbb{R}^n)) \text{ and in } L^p(\tau, \tau + T; L^p(\mathbb{R}^n)), \tag{3.27}$$

$$f(t, \cdot, e^{\alpha z(\theta_t\omega)}v_k) \to \chi \text{ weakly in } L^q(\tau, \tau + T; L^q(\mathbb{R}^n)), \tag{3.28}$$

$$\frac{dv_k}{dt} \to \frac{dv}{dt} \text{ weakly in } L^q(\tau, \tau + T; (V_K \bigcap L^p(\mathbb{R}^n))^*), \quad \forall K \in \mathbb{N}, \tag{3.29}$$

and

$$v_k(\tau + T, \tau, \omega) \to \widetilde{v} \text{ weakly in } L^2(\mathbb{R}^n). \tag{3.30}$$

Note that the embedding $H^s(\mathcal{O}_K) \hookrightarrow L^2(\mathcal{O}_K)$ is compact. Note also that $L^2(\mathcal{O}_K) \hookrightarrow (V_K \bigcap L^p(\mathbb{R}^n))^*$ is continuous. Then by (3.22), (3.25) and the compactness result in [37], after an appropriate diagonal process we find that, up to a subsequence,

$$v_k \to v \text{ strongly in } L^2(\tau, \tau + T; L^2(\mathcal{O}_K)), \quad \forall K \in \mathbb{N}. \tag{3.31}$$

By (3.31) and a diagonal process again, there exists a further subsequence (which is still denoted by $\{v_k\}_{k=1}^\infty$) such that

$$v_k \to v \quad \text{for almost every } (t,x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.32}$$

Since $f$ is continuous, by (3.32) we get

$$f(t,x,e^{\alpha z(\theta_t\omega)}v_k) \to f(t,x,e^{\alpha z(\theta_t\omega)}v) \quad \text{for almost every } (t,x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.33}$$

By (3.24) and (3.33) we infer from Mazur's lemma that

$$f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k) \to f(t,\cdot,e^{\alpha z(\theta_t\omega)}v) \quad \text{weakly in } L^q(\tau,\tau+T;L^q(\mathbb{R}^n)). \tag{3.34}$$

It follows from (3.28) and (3.34) that

$$\chi = f(t,\cdot,e^{\alpha z(\theta_t\omega)}v). \tag{3.35}$$

Now, given $\xi \in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$, denote by

$$\xi_K(x) = \rho(\frac{|x|}{K})\xi(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

By simple computations, one can verify that for each $K \in \mathbb{N}$, $\xi_K \in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$ and

$$\xi_K \to \xi \ \text{ in } \ H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n). \tag{3.36}$$

For every $k > K$ and $\phi \in C_0^\infty(\tau,\tau+T)$, by (3.12)-(3.13) we obtain

$$\begin{aligned} -\int_\tau^{\tau+T}(v_k,\xi_K)\phi' dt + \int_\tau^{\tau+T} a(v_k,\xi_K)\phi dt \\ = \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v_k,\xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k),\xi_K)_{(L^q,L^p)}\phi dt \\ + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(g,\xi_K)\phi dt. \end{aligned} \tag{3.37}$$

Taking the limit of (3) as $k \to \infty$, by (3.26)-(3.28) and (3.35) we get

$$\begin{aligned} -\int_\tau^{\tau+T}(v,\xi_K)\phi' dt + \int_\tau^{\tau+T} a(v,\xi_K)\phi dt \\ = \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,\xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi_K)_{(L^q,L^p)}\phi dt \\ + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(g,\xi_K)\phi dt. \end{aligned} \tag{3.38}$$

Taking the limit of (3.38) as $K \to \infty$, by (3.36) we find for all $\xi \in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$\begin{aligned} -\int_\tau^{\tau+T}(v,\xi)\phi' dt + \int_\tau^{\tau+T} a(v,\xi)\phi dt \\ = \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,\xi)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi)_{(L^q,L^p)}\phi dt \\ + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(g,\xi)\phi dt, \end{aligned}$$

and hence we obtain for all $\xi \in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$\begin{aligned}\frac{d}{dt}(v,\xi)+a(v,\xi)=&\alpha z(\theta_t\omega)(v,\xi)+e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi)_{(L^q,L^p)}\\&+e^{-\alpha z(\theta_t\omega)}(g,\xi)\end{aligned}\tag{3.39}$$

in the sense of distribution on $(\tau,\tau+T)$.

To prove the continuity of $v:[\tau,\infty)\to L^2(\mathbb{R}^n)$, we notice that $v\in L^2(\tau,\tau+T;H^s(\mathbb{R}^n))\bigcap L^p(\tau,\tau+T;L^p(\mathbb{R}^n))$ and $\frac{dv}{dt}\in L^2(\tau,\tau+T;H^{-s}(\mathbb{R}^n))+L^q(\tau,\tau+T;L^q(\mathbb{R}^n))$ by (3.27) and (3.29), respectively. Then by the argument of [37] we infer that $v\in C([\tau,\tau+T],L^2(\mathbb{R}^n))$ and

$$\frac{1}{2}\frac{d}{dt}\|v\|^2=(\frac{dv}{dt},v)_{(H^{-s}+L^q,H^s\bigcap L^p)}\quad\text{for almost every}\quad t\in(\tau,\tau+T).\tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2+a(v,v)=&\alpha z(\theta_t\omega)(v,v)+e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),v)_{(L^q,L^p)}\\&+e^{-\alpha z(\theta_t\omega)}(g,v),\end{aligned}\tag{3.41}$$

which yields the desired energy equation (3.18).

In what follows, we show $v(\tau)=v_\tau$ and $v(\tau+T)=\widetilde{v}$. To that end, we take $\phi\in C^1([\tau,\tau+T])$ and $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$. Similar to (3), by (3.12)-(3.14) we get, for every $k>K$,

$$\begin{aligned}(v_k(\tau+T),\xi_K)\phi(\tau+T)-(v_k(\tau),\xi_K)\phi(\tau)=\int_\tau^{\tau+T}(v_k,\xi_K)\phi' dt-\int_\tau^{\tau+T}a(v_k,\xi_K)\phi dt\\+\alpha\int_\tau^{\tau+T}z(\theta_t\omega)(v_k,\xi_K)\phi dt+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k),\xi_K)_{(L^q,L^p)}\phi dt\\+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(g,\xi_K)\phi dt.\end{aligned}\tag{3.42}$$

As before, by (3.14), (3.26)-(3.28), (3.30) and (3.35) we obtain from (3) that, as $k\to\infty$,

$$\begin{aligned}(\widetilde{v},\xi_K)\phi(\tau+T)-(v_\tau,\xi_K)\phi(\tau)=\int_\tau^{\tau+T}(v,\xi_K)\phi' dt-\int_\tau^{\tau+T}a(v,\xi_K)\phi dt\\+\alpha\int_\tau^{\tau+T}z(\theta_t\omega)(v,\xi_K)\phi dt+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi_K)_{(L^q,L^p)}\phi dt\\+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(g,\xi_K)\phi dt.\end{aligned}$$

As $K\to\infty$ in the above equality, by (3.36) we find that for all $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$\begin{aligned}(\widetilde{v},\xi)\phi(\tau+T)-(v_\tau,\xi)\phi(\tau)=\int_\tau^{\tau+T}(v,\xi)\phi' dt-\int_\tau^{\tau+T}a(v,\xi)\phi dt\\+\alpha\int_\tau^{\tau+T}z(\theta_t\omega)(v,\xi)\phi dt+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi)_{(L^q,L^p)}\phi dt\\+\int_\tau^{\tau+T}e^{-\alpha z(\theta_t\omega)}(g,\xi)\phi dt.\end{aligned}\tag{3.43}$$

On the other hand, by (3.39) we find that the right-hand side of the equality (3.43) is given by

$$(v(\tau+T),\xi)\phi(\tau+T)-(v(\tau),\xi)\phi(\tau)$$

and hence we get

$$(v(\tau+T),\xi)\phi(\tau+T)-(v(\tau),\xi)\phi(\tau)=(\widetilde{v},\xi)\phi(\tau+T)-(v_\tau,\xi)\phi(\tau). \tag{3.44}$$

By choosing $\phi\in C^1([\tau,\tau+T])$ with $\phi(\tau)=1$ and $\phi(\tau+T)=0$, we get from (3.44) that for all $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$(v(\tau),\xi)=(v_\tau,\xi). \tag{3.45}$$

Similarly, by choosing $\phi\in C^1([\tau,\tau+T])$ with $\phi(\tau)=0$ and $\phi(\tau+T)=1$, we get from (3.44) that for all $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$(v(\tau+T),\xi)=(\widetilde{v},\xi). \tag{3.46}$$

By (3.45)-(3.46) we obtain

$$v(\tau)=v_\tau \quad \text{and} \quad v(\tau+T)=\widetilde{v} \quad \text{in} \quad L^2(\mathbb{R}^n), \tag{3.47}$$

which along with (3.30) implies that

$$v_k(\tau+T,\tau,\omega)\to v(\tau+T) \ \text{ weakly in } \ L^2(\mathbb{R}^n). \tag{3.48}$$

Similar to (3.48), one can verify that for every $t\geq\tau$, as $k\to\infty$,

$$v_k(t,\tau,\omega)\to v(t) \ \text{ weakly in } \ L^2(\mathbb{R}^n). \tag{3.49}$$

Note that (3.39) and (3.47) indicate that $v$ is a solution of problem (3.8)-(3.9) in the sense of Definition 3.1, and (3.41) shows that $v$ satisfies the energy equation (3.18).

**Step (iv). Uniqueness and measurability of solutions.** Suppose $v_1$ and $v_2$ are solutions of (3.8)-(3.9). Then for $\widetilde{v}=v_1-v_2$ we have

$$\begin{aligned}&\frac{d\widetilde{v}}{dt}+A(v_1-v_2)\\ &=\alpha z(\theta_t\omega)\widetilde{v}+e^{-\alpha z(\theta_t\omega)}\Big(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_1)-f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_2)\Big)\end{aligned}$$

from which and (3.5) we find that for every $T>0$, there exists $c_1>0$ such that for all $t\in[\tau,\tau+T]$,

$$\frac{d}{dt}\|\widetilde{v}\|^2\leq c_1\|\widetilde{v}\|^2.$$

Then the uniqueness and continuity of solutions in initial data in $L^2(\mathbb{R}^n)$ follows immediately.

Since the solution of problem (3.8)-(3.9) is unique, by (3.49) we see that the whole sequence (not just a subsequence) $v_k(t,\tau,\omega)\to v(t,\tau,\omega)$ weakly in $L^2(\mathbb{R}^n)$ for any $t\geq\tau$ and $\omega\in\Omega$. Because $v_k(t,\tau,\omega)$ is measurable in $\omega\in\Omega$ as proved in [54], we infer that the weak limit $v(t,\tau,\omega)$ is also measurable in $\omega$, which completes the proof. □

Based on Theorem 3.2, we can define a continuous cocycle for problem (3.1)-(3.2). Note that if $v$ is a solution of (3.8)-(3.9), then by (3.7) we see that $u$ is a solution of (3.1)-(3.2) where $u$ is given by

$$u(t,\tau,\omega,u_\tau)=e^{\alpha z(\theta_t\omega)}v(t,\tau,\omega,v_\tau)$$

with $u_\tau=e^{\alpha z(\theta_\tau\omega)}v_\tau$. Define a mapping $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ such that for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $u_\tau\in L^2(\mathbb{R}^n)$,

$$\Phi(t,\tau,\omega,u_\tau)=u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau)=e^{\alpha z(\theta_t\omega)}v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau), \tag{3.50}$$

where $v_\tau = e^{-\alpha z(\omega)} u_\tau$. It follows from Theorem 3.2 that $\Phi$ is a continuous cocycle in $L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$.The main purpose of this paper is to prove the existence of attractors of $\Phi$ in $L^2(\mathbb{R}^n)$. To that end, we recall a family of bounded nonempty subsets of $L^2(\mathbb{R}^n)$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, is tempered if for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{ct} \|D(\tau + t, \theta_t\omega)\| = 0,$$

where the notation $\|D\|$ for a subset $D$ of $L^2(\mathbb{R}^n)$ is understood as $\|D\| = \sup_{u\in D} \|u\|$. The collection of all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^n)$ is denoted $\mathcal{D}$, that is,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \ D \ \text{ is tempered in } \ L^2(\mathbb{R}^n)\}. \tag{3.51}$$

In this case, a $\mathcal{D}$-pullback attractor is also called a tempered attractor since $\mathcal{D}$ given by (3.51) contains all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^n)$.

From now on, we assume that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{0} e^{\lambda s} \left( \|g(s+\tau,\cdot)\|^2 + \|\psi_1(s+\tau,\cdot)\|_{L^1(\mathbb{R}^n)} \right) ds < \infty. \tag{3.52}$$

When deriving the existence of tempered pullback absorbing sets, we will further assume that $g$ and $\psi_1$ are tempered in the sense that for every $c > 0$,

$$\lim_{r\to-\infty} e^{cr} \int_{-\infty}^{0} e^{\lambda s} \left( \|g(s+r,\cdot)\|^2 + \|\psi_1(s+r,\cdot)\|_{L^1(\mathbb{R}^n)} \right) ds = 0. \tag{3.53}$$

It is clear that (3.52) and (3.53) do not imply that $g$ is bounded in $L^2(\mathbb{R}^n)$ when $t \to \infty$.

## 4. Uniform estimates of solutions.

In this section, we derive uniform estimates on the solutions of the non-local fractional stochastic equations in $H^s(\mathbb{R}^n)$ as well as the uniform estimates on the tails of solutions for large space and time variables. The estimates in $L^2(\mathbb{R}^n)$ are given below.

**Lemma 4.1.** *Under conditions (3.3)-(3.5) and (3.52), for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\sigma) > 0$ such that for all $t \geq T$, the solution $v$ of system (3.8)-(3.9) satisfies*

$$\begin{aligned}
&\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \int_{-t}^{\sigma-\tau} A(s) \|v(s+\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)} ds \\
&\quad + 2\beta \int_{-t}^{\sigma-\tau} A(s) e^{(p-2)\alpha z(\theta_s\omega)} \|v(s+\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^p_{L^p(\mathbb{R}^n)} ds \\
&\leq M_1 + M_1 \int_{-\infty}^{\sigma-\tau} A(s) e^{-2\alpha z(\theta_s\omega)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1(\mathbb{R}^n)}) ds,
\end{aligned}$$

*where $A(s) = e^{\frac{5}{4}\lambda(s-\sigma+\tau) - 2\alpha \int_{\sigma-\tau}^{s} z(\theta_r\omega) dr}$, $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $M_1$ is a positive number independent of $\tau$, $\omega$ and $D$.*

*Proof.* The proof is similar to the case of bounded domains as in [54]. For the reader's convenience, we here sketch the main idea. First, by (3.3) and (3.18) we have

$$\frac{d}{dt} \|v(t,\tau,\omega,v_\tau)\|^2 + 2\lambda \|v\|^2 + C(n,s) \|v\|^2_{\dot{H}^s} + 2\beta e^{(p-2)\alpha z(\theta_t\omega)} \int_{\mathbb{R}^n} |v|^p dx$$

$$\leq 2\alpha z(\theta_t\omega)\|v\|^2 + 2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} g(t,x)vdx + 2e^{-2\alpha z(\theta_t\omega)}\|\psi_1(t)\|_{L^1(\mathbb{R}^n)}. \quad (4.1)$$

Note that the Young inequality implies

$$2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} g(t,x)vdx \leq \frac{1}{4}\lambda\|v\|^2 + 4\lambda^{-1}e^{-2\alpha z(\theta_t\omega)}\|g(t)\|^2,$$

which along with (4.1) yields

$$\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2 + (\frac{5}{4}\lambda - 2\alpha z(\theta_t\omega))\|v\|^2 + \frac{1}{2}\lambda\|v\|^2$$
$$+C(n,s)\|v\|^2_{\dot{H}^s} + 2\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v|^p dx$$
$$\leq 4\lambda^{-1}e^{-2\alpha z(\theta_t\omega)}\|g(t)\|^2 + 2e^{-2\alpha z(\theta_t\omega)}\|\psi_1(t)\|_{L^1(\mathbb{R}^n)}. \quad (4.2)$$

Solving (4.2) for $\|v\|^2$ on the interval $(\tau-t,\sigma)$ by introducing the integrating factor $e^{\frac{5}{4}\lambda t - 2\alpha\int_0^t z(\theta_r\omega)dr}$, and then replacing $\omega$ by $\theta_{-\tau}\omega$ we obtain

$$\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{\lambda}{2}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds$$
$$+C\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot{H}^s} ds$$
$$+2\beta\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{(p-2)\alpha z(\theta_{s-\tau}\omega)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^p_{L^p(\mathbb{R}^n)} ds$$
$$\leq e^{\frac{5}{4}\lambda(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t} z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t}\|^2$$
$$+4\lambda^{-1}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|g(s)\|^2 ds$$
$$+2\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds. \quad (4.3)$$

Since $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ with $D \in \mathcal{D}$, by (3.6) one can verify that

$$\lim_{t\to\infty} e^{\frac{5}{4}\lambda(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t} z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t}\|^2 = 0. \quad (4.4)$$

On the other hand, by (3.52) and (3.6) we find that for all $\sigma \geq \tau - t$,

$$\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|g(s)\|^2 ds$$
$$\leq \int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|g(s)\|^2 ds < \infty \quad (4.5)$$

and

$$\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds$$
$$\leq \int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds < \infty. \quad (4.6)$$

It follows from (4.3)-(4) that there exists $T = T(\tau,\omega,D,\alpha) > 0$ such that for all $t \geq T$,

$$\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{\lambda}{2}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds$$
$$+C(n,s)\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot{H}^s} ds$$

$$+2\beta \int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{(p-2)\alpha z(\theta_{s-\tau}\omega)} \|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^p_{L^p(\mathbb{R}^n)} ds$$

$$\le 1 + 4\lambda^{-1} \int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)} \|g(s)\|^2 ds$$

$$+ 2 \int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)} \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds. \quad (4.7)$$

After changing of variables, the desired estimates follows from (4.7) immediately. □

As a consequence of Lemma 4.1, we see that problem (3.8)-(3.9) has a tempered pullback absorbing set in $L^2(\mathbb{R}^n)$.

**Corollary 4.2.** *Under conditions (3.3)-(3.5) and (3.53), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\alpha) > 0$ such that the solution $v$ of (3.8)-(3.9) with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ satisfies, for all $t \ge T$,*

$$v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \in B(\tau,\omega), \quad (4.8)$$

*where $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is given by*

$$B(\tau,\omega) = \{v \in L^2(\mathbb{R}^n) : \|v\|^2 \le R(\tau,\omega)\},$$

*with $R = R(\tau,\omega)$ being a positive number given by*

$$R = M_1 + M_1 \int_{-\infty}^{0} e^{\frac{5}{4}\lambda s-2\alpha\int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1(\mathbb{R}^n)}) ds. \quad (4.9)$$

*Moreover, $R = \{R(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered in the sense that for any $c > 0$,*

$$\lim_{t\to\infty} e^{-ct} R(\tau-t,\theta_{-t}\omega) = 0. \quad (4.10)$$

*Proof.* (4.8) follows from Lemma 4.1 when $\sigma = \tau$, and the convergence of (4.10) can be proved in the same way as in the case of bounded domains which can be found in [54]. The details are omitted here. □

Next, we derive uniform estimates of solutions in $H^s(\mathbb{R}^n)$ for which we further assume that the function $\psi_4$ in (3.5) belongs to $L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))$ and the nonlinearity $f$ satisfies, for all $t, u \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$,

$$|f(t,x,u) - f(t,y,u)| \le |\psi_5(x) - \psi_5(y)| \quad (4.11)$$

where $\psi_5 \in H^s(\mathbb{R}^n)$.

**Lemma 4.3.** *Under conditions (3.3)-(3.5), (4.11) and (3.52), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\alpha) > 0$ such that for any $t \ge T$, the solution $v$ of problem (3.8)-(3.9) with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ satisfies*

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}$$

$$\le M_2 + M_2 \int_{-\infty}^{0} e^{\frac{5}{4}\lambda s-2\alpha\int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} (1 + \|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1(\mathbb{R}^n)}) ds,$$

*where $M_2$ is a positive number independent of $\tau$, $\omega$ and $D$.*

*Proof.* Multiplying (3.8) by $(-\Delta)^s v$, we obtain

$$\begin{aligned}\frac{d}{dt}&\|(-\Delta)^{\frac{s}{2}}v\|^2+2\|(-\Delta)^s v\|^2+2(\lambda-\alpha z(\theta_t\omega))\|(-\Delta)^{\frac{s}{2}}v\|^2\\&=2e^{-\alpha z(\theta_t\omega)}\left(f(t,x,e^{\alpha z(\theta_t\omega)}v),(-\Delta)^s v\right)+2e^{-\alpha z(\theta_t\omega)}(g(t,x),(-\Delta)^s v).\end{aligned}\tag{4.12}$$

We now estimate the right-hand side of (4.12). For the first term, by (3.5) and (4.11) we have with $C_{ns}=C(n,s)$

$$\begin{aligned}&2e^{-\alpha z(\theta_t\omega)}\left(f(t,x,e^{\alpha z(\theta_t\omega)}v),(-\Delta)^s v\right)\\&=2e^{-\alpha z(\theta_t\omega)}\left((-\Delta)^{\frac{s}{2}}f(t,x,e^{\alpha z(\theta_t\omega)}v),(-\Delta)^{\frac{s}{2}}v\right)\\&=C_{ns}e^{-\alpha z(\theta_t\omega)}\left(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),v\right)_{\dot H^s(\mathbb{R}^n)}\\&=C_{ns}e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(f(t,x,e^{\alpha z(\theta_t\omega)}v(x))-f(t,y,e^{\alpha z(\theta_t\omega)}v(y))\right)B(x,y)dxdy\\&=C_{ns}e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(f(t,x,e^{\alpha z(\theta_t\omega)}v(x))-f(t,y,e^{\alpha z(\theta_t\omega)}v(x))\right)B(x,y)dxdy\\&+C_{ns}e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(f(t,y,e^{\alpha z(\theta_t\omega)}v(x))-f(t,y,e^{\alpha z(\theta_t\omega)}v(y))\right)B(x,y)dxdy\\&\le C_{ns}e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\psi_5(x)-\psi_5(y)||v(x)-v(y)|}{|x-y|^{n+2s}}dxdy\\&+C_{ns}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\psi_4(t,y)(v(x)-v(y))^2}{|x-y|^{n+2s}}dxdy\\&\le C_{ns}e^{-\alpha z(\theta_t\omega)}\|\psi_5\|_{\dot H^s(\mathbb{R}^n)}\|v\|_{\dot H^s(\mathbb{R}^n)}+C_{ns}\|\psi_4\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))}\|v\|^2_{\dot H^s(\mathbb{R}^n)}\\&\le\frac12 C_{ns}e^{-2\alpha z(\theta_t\omega)}\|\psi_5\|^2_{H^s(\mathbb{R}^n)}+(\frac12+\|\psi_4\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))})C_{ns}\|v\|^2_{\dot H^s(\mathbb{R}^n)}\\&\le\frac12 C_{ns}e^{-2\alpha z(\theta_t\omega)}\|\psi_5\|^2_{H^s(\mathbb{R}^n)}+(1+2\|\psi_4\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))})\|(-\Delta)^{\frac{s}{2}}v\|^2,\end{aligned}\tag{4.13}$$

where

$$B(x,y)=\frac{v(x)-v(y)}{|x-y|^{n+2s}}.$$

For the last term on the right-hand side of (4.12), we have

$$|2e^{-\alpha z(\theta_t\omega)}(g(t,x),(-\Delta)^s v)|\le\frac12\|(-\Delta)^s v\|^2+2e^{-2\alpha z(\theta_t\omega)}\|g(t)\|^2.\tag{4.14}$$

It follows from (4.12)-(4.14) that

$$\begin{aligned}\frac{d}{dt}&\|(-\Delta)^{\frac{s}{2}}v\|^2+\|(-\Delta)^s v\|^2+2(\lambda-\alpha z(\theta_t\omega))\|(-\Delta)^{\frac{s}{2}}v\|^2\\&\le c_1\|(-\Delta)^{\frac{s}{2}}v\|^2+\left(2\|g(t)\|^2+c_2\right)e^{-2\alpha z(\theta_t\omega)}.\end{aligned}\tag{4.15}$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, let $s\in(\tau-1,\tau)$. Multiplying (4.15) by $e^{\int_0^t(\frac54\lambda-2\alpha z(\theta_s\omega))ds}$ and integrating over $(s,\tau)$, we infer that

$$\|(-\Delta)^{\frac{s}{2}}v(\tau,\tau-t,\omega,v_{\tau-t})\|^2\le e^{\int_\tau^s(\frac54\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{s}{2}}v(s,\tau-t,\omega,v_{\tau-t})\|^2$$

$$\begin{aligned} &+\int_s^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}c_1\|(-\Delta)^{\frac{s}{2}}v(\varsigma,\tau-t,\omega,v_{\tau-t})\|^2d\varsigma \\ &+\int_s^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}(2\|g(\varsigma)\|^2+c_2)e^{-2\alpha z(\theta_\varsigma\omega)}d\varsigma. \end{aligned} \tag{4.16}$$

Integrating again with respect to $s$ on $(\tau-1,\tau)$, we obtain

$$\begin{aligned} &\|(-\Delta)^{\frac{s}{2}}v(\tau,\tau-t,\omega,v_{\tau-t})\|^2 \\ &\le\int_{\tau-1}^\tau e^{\int_\tau^s(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{s}{2}}v(s,\tau-t,\omega,v_{\tau-t})\|^2ds \\ &+\int_{\tau-1}^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}c_1\|(-\Delta)^{\frac{s}{2}}v(\varsigma,\tau-t,\omega,v_{\tau-t})\|^2d\varsigma \\ &+\int_{\tau-1}^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}(2\|g(\varsigma)\|^2+c_2)e^{-2\alpha z(\theta_\varsigma\omega)}d\varsigma. \end{aligned} \tag{4.17}$$

Substituting $\theta_{-\tau}\omega$ for $\omega$, then we deduce from (4.17) that,

$$\begin{aligned} &\|(-\Delta)^{\frac{s}{2}}v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 \\ &\le\int_{\tau-1}^\tau e^{\int_\tau^s(\frac{5}{4}\lambda-2\alpha z(\theta_{\xi-\tau}\omega))d\xi}\|(-\Delta)^{\frac{s}{2}}v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2ds \\ &+\int_{\tau-1}^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_{\xi-\tau}\omega))d\xi}c_1\|(-\Delta)^{\frac{s}{2}}v(\varsigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2d\varsigma \\ &+\int_{\tau-1}^\tau e^{\int_\tau^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_{\xi-\tau}\omega))d\xi}(2\|g(\varsigma)\|^2+c_2)e^{-2\alpha z(\theta_{\varsigma-\tau}\omega)}d\varsigma \\ &\le\int_{-1}^0 e^{\int_0^s(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{s}{2}}v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2ds \\ &+\int_{-1}^0 e^{\int_0^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}c_1\|(-\Delta)^{\frac{s}{2}}v(\varsigma+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2d\varsigma \\ &+\int_{-1}^0 e^{\int_0^\varsigma(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}(2\|g(\varsigma+\tau)\|^2+c_2)e^{-2\alpha z(\theta_\varsigma\omega)}d\varsigma \end{aligned} \tag{4.18}$$

which along with Lemma 4.1 for $\sigma=\tau$ implies the desired estimates. $\square$

To prove the pullback asymptotic compactness of the cocycle associated with the problem (3.8)-(3.9) on the unbounded domain $\mathbb{R}^n$, we need to derive the uniform estimates on the tail parts of the solutions for large space variables when time is large enough.

**Lemma 4.4.** *Suppose that (3.3)-(3.5) and (3.52) hold. Then for every $\varepsilon>0$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T(\tau,\omega,D,\varepsilon,\alpha)>0$, $K=K(\tau,\omega,\varepsilon)\ge1$ such that for all $t\ge T$ and $k\ge K$, the solution $v$ of problem (3.8)-(3.9) with $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ satisfies*

$$\int_{|x|\ge k}|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})(x)|^2dx\le\varepsilon.$$

*Proof.* Let $\chi(s)=1-\rho(s)$ for all $0\le s<\infty$ where $\rho$ is the smooth function given by (3.11). Then we find that $0\le\chi(s)\le1$ for all $s\ge0$ and

$$\chi(s)=\begin{cases}0, & \text{if } 0\le s\le\frac{1}{2},\\ 1, & \text{if } s\ge1.\end{cases} \tag{4.19}$$

Note that there exists a positive constant $c$ such that $|\chi'(s)| \le c$ for all $s \ge 0$. Multiplying (3.8) by $\chi(\frac{|x|}{k})v$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right)|v|^2dx + (\lambda - \alpha z(\theta_t\omega))\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v|^2dx \\
&\qquad + \int_{\mathbb{R}^n}(-\Delta)^s v\ \chi\left(\frac{|x|}{k}\right)vdx \\
&= e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} f(t,x,e^{\alpha z(\theta_t\omega)}v)\chi\left(\frac{|x|}{k}\right)vdx + e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} g(t,x)\ \chi\left(\frac{|x|}{k}\right)vdx.
\end{aligned} \tag{4.20}$$

For the third term on the left-hand side of (4.20), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}(-\Delta)^s v\ \chi\left(\frac{|x|}{k}\right)vdx = -\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{2}}v\ (-\Delta)^{\frac{s}{2}}\left(\chi\left(\frac{|x|}{k}\right)v\right)dx \\
&= -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\chi\left(\frac{|x|}{k}\right)v(x) - \chi\left(\frac{|y|}{k}\right)v(y)\right)(v(x)-v(y))}{|x-y|^{n+2s}}dxdy \\
&= -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\chi\left(\frac{|x|}{k}\right)(v(x)-v(y))^2}{|x-y|^{n+2s}}dxdy \\
&-\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right)(v(x)-v(y))v(y)}{|x-y|^{n+2s}}dxdy \\
&\le -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right)(v(x)-v(y))v(y)}{|x-y|^{n+2s}}dxdy.
\end{aligned} \tag{4.21}$$

Note that the right-hand side of (4.21) is controlled by

$$\begin{aligned}
&\frac{1}{2}C(n,s)\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right)(v(x)-v(y))v(y)}{|x-y|^{n+2s}}\right|dxdy \\
&\le \frac{1}{2}C(n,s)\int_{\mathbb{R}^n}|v(y)|\left(\int_{\mathbb{R}^n}\frac{\left|\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right||v(x)-v(y)|}{|x-y|^{n+2s}}dx\right)dy \\
&\le \frac{1}{2}C(n,s)\|v\|\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{\left|\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right||v(x)-v(y)|}{|x-y|^{n+2s}}dx\right)^2dy\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}C(n,s)\|v\|\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{\left|\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{n+2s}}dx\ \int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dx\right)dy\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.22}$$

We now estimate the first integral in (4.22). Given $y \in \mathbb{R}$, for $z = \frac{x}{k}$ we have

$$\int_{\mathbb{R}^n}\frac{\left|\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{n+2s}}dx = \frac{1}{k^{2s}}\int_{\mathbb{R}^n}\frac{\left|\chi(|z|) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|z-\frac{y}{k}|^{n+2s}}dz$$

$$
\begin{aligned}
&= \frac{1}{k^{2s}} \int_{\mathbb{R}^n} \frac{\left|\chi(|\xi + \frac{y}{k}|) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|\xi|^{n+2s}} d\xi \\
&= \frac{1}{k^{2s}} \int_{|\xi|\le 1} \frac{\left|\chi(|\xi + \frac{y}{k}|) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|\xi|^{n+2s}} d\xi + \frac{1}{k^{2s}} \int_{|\xi|>1} \frac{\left|\chi(|\xi + \frac{y}{k}|) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|\xi|^{n+2s}} d\xi \\
&\le \frac{1}{k^{2s}} \int_{|\xi|\le 1} \frac{|\chi'(r)|^2 \ \left||\xi + \frac{y}{k}| - |\frac{y}{k}|\right|^2}{|\xi|^{n+2s}} d\xi + \frac{4}{k^{2s}} \int_{|\xi|>1} \frac{1}{|\xi|^{n+2s}} d\xi \\
&\le \frac{c_1}{k^{2s}} \int_{|\xi|\le 1} \frac{1}{|\xi|^{n+2s-2}} d\xi + \frac{4}{k^{2s}} \int_{|\xi|>1} \frac{1}{|\xi|^{n+2s}} d\xi.
\end{aligned}
$$

Since $s \in (0,1)$, we see that the above integrals are convergent. Thus we get

$$
\int_{\mathbb{R}^n} \frac{\left|\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{n+2s}} dx \le \frac{c_2}{k^{2s}}, \tag{4.23}
$$

where $c_2$ is a positive constant independent of $k \in \mathbb{N}$ and $y \in \mathbb{R}^n$ By (4.22)-(4.23) we get

$$
\begin{aligned}
&\frac{1}{2} C(n,s) \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(\chi\left(\frac{|x|}{k}\right) - \chi\left(\frac{|y|}{k}\right)\right)(v(x) - v(y)) v(y)}{|x-y|^{n+2s}} \right| dxdy \\
&\le \frac{1}{2} C(n,s) \sqrt{c_2}\ k^{-s} \|v\| \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}} \\
&\le \frac{1}{2} C(n,s) \sqrt{c_2}\ k^{-s} \|v\|^2_{H^s(\mathbb{R}^n)},
\end{aligned}
$$

which together with (4.21) implies that

$$
- \int_{\mathbb{R}^n} (-\Delta)^s v \ \chi\left(\frac{|x|}{k}\right) v dx \le \frac{1}{2} C(n,s) \sqrt{c_2}\ k^{-s} \|v\|^2_{H^s(\mathbb{R}^n)}. \tag{4.24}
$$

For the first term on the right-hand side of (4.20), by (3.3) one has

$$
\begin{aligned}
&e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} f(t,x,e^{\beta z(\theta_t \omega)} v) \chi\left(\frac{|x|}{k}\right) v dx \\
&\le -\beta e^{-2\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} |e^{\alpha z(\theta_t \omega)} v|^p \chi\left(\frac{|x|}{k}\right) dx + e^{-2\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} \psi_1(t,x) \chi\left(\frac{|x|}{k}\right) dx \\
&\le e^{-2\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} |\psi_1(t,x)| \chi\left(\frac{|x|}{k}\right) dx.
\end{aligned} \tag{4.25}
$$

For the second term on the right-hand side of (4.20), one has

$$
\begin{aligned}
e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} g(t,x) \chi\left(\frac{|x|}{k}\right) v dx \le \frac{e^{-2\alpha z(\theta_t \omega)}}{\lambda} \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |g(t,x)|^2 dx \\
+ \frac{\lambda}{4} \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v|^2 dx.
\end{aligned} \tag{4.26}
$$

Substituting (4.24)-(4.26) into (4.20), we deduce

$$
\frac{d}{dt} \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v|^2 dx + (\frac{3}{2}\lambda - 2\alpha z(\theta_t \omega)) \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v|^2 dx
$$

$$\le c_3 k^{-s}\|v\|^2_{H^s(\mathbb{R}^n)} + 2e^{-2\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|\psi_1(t,x)|dx$$
$$+\frac{2}{\lambda}e^{-2\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|g(t,x)|^2dx. \tag{4.27}$$

Since $s\in(0,1)$ and $c_3$ is independent of $k$, for any given $\varepsilon>0$, there exists $K_1=K_1(\varepsilon)\ge 1$ such that for all $k\ge K_1$,

$$c_3k^{-s}\|v\|^2_{H^s(\mathbb{R}^n)}\le\varepsilon\|v\|^2_{H^s(\mathbb{R}^n)}. \tag{4.28}$$

On the other hand, by the definition of $\chi$ we find that

$$\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)(|g(t,x)|^2+|\psi_1(t,x)|)dx\le\int_{|x|\ge\frac12 k}(|g(t,x)|^2+|\psi_1(t,x)|)dx. \tag{4.29}$$

By (4)-(4.29) we get, for all $k\ge K_1$,

$$\frac{d}{dt}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v|^2dx+(\frac32\lambda-2\alpha z(\theta_t\omega))\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v|^2dx$$
$$\le\varepsilon\|v\|^2_{H^s(\mathbb{R}^n)}+c_4e^{-2\alpha z(\theta_t\omega)}\int_{|x|\ge\frac12 k}(|g(t,x)|^2+|\psi_1(t,x)|)dx. \tag{4.30}$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, multiplying (4) by $e^{\int_0^t(\frac54\lambda-2\alpha z(\theta_r\omega))dr}$ and integrating over $(\tau-t,\tau)$, we get, for all $k\ge K_1$,

$$\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v(\tau,\tau-t,\omega,v_{\tau-t})|^2dx-e^{\int_\tau^{\tau-t}(\frac54\lambda-2\alpha z(\theta_r\omega))dr}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v_{\tau-t}(x)|^2dx$$
$$\le\varepsilon\int_{\tau-t}^{\tau}e^{\int_\tau^s(\frac54\lambda-2\alpha z(\theta_r\omega))dr}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}ds$$
$$+c_4\int_{\tau-t}^{\tau}\int_{|x|\ge\frac12 k}e^{\int_\tau^s(\frac54\lambda-2\alpha z(\theta_r\omega))dr}e^{-2\alpha z(\theta_s\omega)}(g^2(s,x)+|\psi_1(s,x)|)dxds. \tag{4.31}$$

Replacing $\omega$ by $\theta_{-\tau}\omega$ in (4) we get, for all $k\ge K_1$,

$$\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx$$
$$-e^{\int_\tau^{\tau-t}(\frac54\lambda-2\alpha z(\theta_{r-\tau}\omega))dr}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v_{\tau-t}(x)|^2dx$$
$$\le\varepsilon\int_{\tau-t}^{\tau}e^{\int_\tau^s(\frac54\lambda-2\alpha z(\theta_{r-\tau}\omega))dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}ds$$
$$+c_4\int_{\tau-t}^{\tau}\int_{|x|\ge\frac12 k}e^{\int_\tau^s(\frac54\lambda-2\alpha z(\theta_{r-\tau}\omega))dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}(g^2(s,x)+|\psi_1(s,x)|)dxds. \tag{4.32}$$

After changing of variables in (4.32), we obtain, for all $k\ge K_1$,

$$\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx$$
$$\le e^{\int_0^{-t}(\frac54\lambda-2\alpha z(\theta_r\omega))dr}\int_{\mathbb{R}^n}\chi\left(\frac{|x|}{k}\right)|v_{\tau-t}(x)|^2dx$$
$$+\varepsilon\int_{-t}^{0}e^{\int_0^s(\frac54\lambda-2\alpha z(\theta_r\omega))dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}ds$$

$$+c_4 \int_{-t}^{0} \int_{|x|\geq \frac{1}{2}k} e^{\int_0^s (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} e^{-2\alpha z(\theta_s \omega)} (g^2(s+\tau, x) + |\psi_1(s+\tau, x)|) dx ds. \tag{4.33}$$

We now estimate the right-hand side of (4). For the first term, since $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ we have

$$e^{\int_0^{-t} (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v_{\tau-t}(x)|^2 dx$$

$$\leq e^{-\frac{5}{4}\lambda t - 2\alpha \int_0^{-t} z(\theta_r \omega)dr} \|v_{\tau-t}\|^2 \leq e^{-\frac{5}{4}\lambda t - 2\alpha \int_0^{-t} z(\theta_r \omega)dr} e^{-2\alpha z(\theta_{-t}\omega)} \|D(\tau - t, \theta_{-t}\omega)\|^2.$$

By (3.6) and the fact that $D \in \mathcal{D}$ we find that the right-hand side of the above inequality converges to zero as $t \to \infty$, and hence there exists $T_1 = T_1(\tau, \omega, D, \varepsilon) > 0$ such that for all $t \geq T_1$,

$$e^{\int_0^{-t} (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v_{\tau-t}(x)|^2 dx \leq \varepsilon. \tag{4.34}$$

By Lemma 4.1 with $\sigma = \tau$, we see that there exists $T_2 = T_2(\tau, \omega, D, \varepsilon) \geq T_1$ such that for all $t \geq T_2$,

$$\varepsilon \int_{-t}^{0} e^{\int_0^s (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} \|v(s+\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2_{H^s(\mathbb{R}^n)} ds \leq \varepsilon R(\tau, \omega), \tag{4.35}$$

where $R(\tau, \omega)$ is the number as in (4.9). For the last term in (4), by (3.6) and (3.52) we find that

$$\int_{-\infty}^{0} \int_{\mathbb{R}^n} e^{\int_0^s (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} e^{-2\alpha z(\theta_s \omega)} (g^2(s+\tau, x) + |\psi_1(s+\tau, x)|) < \infty,$$

and hence there exists $K_2 = K_2(\tau, \omega, \varepsilon) \geq K_1$ such that for all $k \geq K_2$,

$$c_4 \int_{-\infty}^{0} \int_{|x|\geq \frac{1}{2}k} e^{\int_0^s (\frac{5}{4}\lambda - 2\alpha z(\theta_r \omega))dr} e^{-2\alpha z(\theta_s \omega)} (g^2(s+\tau, x) + |\psi_1(s+\tau, x)|) \leq \varepsilon. \tag{4.36}$$

It follows from (4)-(4.36) that for all $k \geq K_2$ and $t \geq T_2$,

$$\int_{|x|\geq k} |v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx$$

$$\leq \int_{\mathbb{R}^n} \chi\left(\frac{|x|}{k}\right) |v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx \leq \varepsilon(2 + R(\tau, \omega)).$$

This completes the proof. □

5. **Existence of random attractors.** In this section, we prove the existence and uniqueness of tempered pullback attractors for the non-local fractional stochastic equation (3.1)-(3.2). To that end, we need to establish the existence of tempered random absorbing sets and the pullback asymptotic compactness of the cocycle $\Phi$.

**Lemma 5.1.** *Under conditions (3.3)-(3.5) and (3.53), the cocycle $\Phi$ has a closed measurable pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ where for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the set $K(\tau, \omega)$ is defined by*

$$K(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq e^{2\alpha z(\omega)} R(\tau, \omega)\},$$

*where $R(\tau, \omega)$ is the same number as in (4.9).*

*Proof.* First, by (3.6) and (4.10) we see that $K \in \mathcal{D}$, that is, for every $c > 0$,

$$\lim_{t\to\infty} e^{-ct}\|K(\tau - t, \theta_{-t}\omega)\| = 0.$$

On the other hand, by (3.7) we get

$$u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) = e^{\alpha z(\omega)} v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) \quad \text{with} \quad u_{\tau-t} = e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t}. \tag{5.1}$$

Then it follows from Corollary 4.2 that for any $D \in \mathcal{D}$ and $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that for all $t \geq T$,

$$v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) \in B(\tau, \omega), \tag{5.2}$$

where $B(\tau, \omega)$ is the same set as in (4.8). By (5.1)-(5.2) we find that for all $t \geq T$,

$$u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \in K(\tau, \omega),$$

which along with (3.50) implies that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, u_{\tau-t}) \in K(\tau, \omega).$$

This shows that $K$ is a $\mathcal{D}$-pullback absorbing set of $\Phi$. It is clear that $R(\tau, \omega)$ is measurable in $\omega \in \Omega$, which implies the measurability of $K(\tau, \omega)$ in $\omega \in \Omega$. □

The uniform estimates of the solutions of problem (3.1)-(3.2) in $H^s(\mathbb{R}^n)$ is given below.

**Lemma 5.2.** *Under conditions (3.3)-(3.5), (4.11) and (3.52), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that for any $t \geq T$, the solution $u$ of problem (3.1)-(3.2) with $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ satisfies*

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}$$

$$\leq M_3 + M_3 \int_{-\infty}^{0} e^{\frac{5}{4}\lambda s - 2\alpha \int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} (1 + \|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1(\mathbb{R}^n)}) ds,$$

*where $M_3 = M_2 e^{2\alpha z(\omega)}$ and $M_2$ is the positive number as in Lemma 4.3.*

*Proof.* This estimate follows from (5.1) and Lemma 4.3 directly. □

Based on Lemma 4.4, one can derive the uniform estimates on the tails of solutions of problem (3.1)-(3.2) as stated below.

**Lemma 5.3.** *Suppose that (3.3)-(3.5) and (3.52) hold. Then for every $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \varepsilon, \alpha) > 0$, $K = K(\tau, \omega, \varepsilon) \geq 1$ such that for all $t \geq T$ and $k \geq K$, the solution $u$ of problem (3.1)-(3.2) with $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ satisfies*

$$\int_{|x|\geq k} |u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})(x)|^2 dx \leq \varepsilon.$$

*Proof.* This is an immediate consequence of Lemma 4.4 together with the arguments of the proof of Lemma 5.1. The details are omitted here. □

The next lemma is concerned with the $\mathcal{D}$-pullback asymptotic compactness of $\Phi$.

**Lemma 5.4.** *Under conditions (3.3)-(3.5), (4.11) and (3.53), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$ whenever $t_n \to \infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$.*

*Proof.* By (3.50) we have

$$\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}),$$

which along with Lemma 5.3 shows that for every $\varepsilon > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $K = K(\tau, \omega, \varepsilon) \geq 1$ and $N_1 = N_1(\tau, \omega, D, \varepsilon) \geq 1$ such that for all $n \geq N_1$,

$$\|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})\|_{L^2(|x|\geq K)} \leq \frac{\varepsilon}{2}. \tag{5.3}$$

By Lemma 5.2 we find that there exists $N_2 = N_2(\tau, \omega, D, \varepsilon) \geq N_1$ such that for all $n \geq N_2$,

$$\|\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\|_{H^s(|x|<K)} \leq L(\tau, \omega),$$

where $L(\tau, \omega)$ is a positive constant. Since $s \in (0, 1)$, the embedding $H^s(|x| < K) \hookrightarrow L^2(|x| < K)$ is compact, which together with (5.3) implies $\{\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ has a finite covering in $L^2(\mathbb{R}^n)$ of balls of radii less than $\varepsilon$. As a consequence, we infer that the sequence $\{\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ is precompact in $L^2(\mathbb{R}^n)$. □

We now present our main result of this paper as follows.

**Theorem 5.5.** *Suppose (3.3)-(3.5), (4.11) and (3.53) hold. Then the cocycle $\Phi$ of problem (3.1)-(3.2) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathbb{R}^n)$.*

*Proof.* This is an immediate consequence of Lemmas 5.1,5.4 and Proposition 2.4. □

Regarding the periodicity of $\mathcal{D}$-pullback attractors, we have the following result.

**Theorem 5.6.** *Let (3.3)-(3.5), (4.11) and (3.53) hold. Assume further that for each fixed $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, the functions $f(t, x, s)$, $g(t, x)$ and $\psi_1(t, x)$ are $T$-periodic in $t \in \mathbb{R}$. Then the $\mathcal{D}$-pullback attractor $\mathcal{A}$ of $\Phi$ is also $T$-periodic, that is, $\mathcal{A}(\tau + T, \omega) = \mathcal{A}(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.*

*Proof.* Since $f(t, x, s)$ and $g(t, x)$ are $T$-periodic in $t \in \mathbb{R}$, we find that the cocycle $\Phi$ is also $T$-periodic. Since $g(t, x)$ and $\psi_1(t, x)$ are $T$-periodic in $t \in \mathbb{R}$, by Lemma 5.1 we see that the $\mathcal{D}$-pullback absorbing set $K$ is also $T$-periodic. Then the $T$-periodicity of $\mathcal{A}$ follows from Proposition 2.5 immediately. □

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